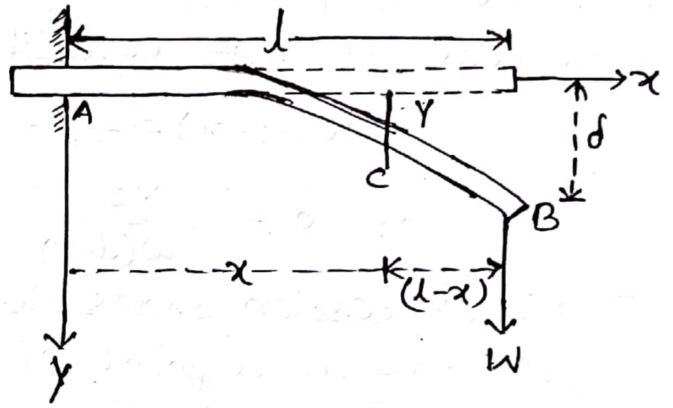
Prof. Vidya Sagar
Department of Physics
B.Sc (Part-I) Hons

Expression for the depression produced in case of a light Cantilever fixed at one end and loaded at the other end.

Ans:- A beam is a rigid body of a uniform rectangular transverse section throughout its length. When a beam is clamped at one end and the loaded at the other end it is specially called Cantilever.



Let us consider a thin, uniform and light beam of length l clamped horizontally at one end A. When it is loaded with a weight W at the free end B, the end B is depressed downward compared to A so that the beam undergoes bending. Since the beam is light, the whole depression may be taken as due to to the load W.

Lets us a section of the beam at C, distant x from A, and consider the equilibrium of the part CB. Since the beam is fixed at A, the load W at B exerts an external torque tending to rotate it clockwise. Its magnitude is $W(l-x)$. This torque is balanced by an anti-clockwise ~~Its magnitude~~ restoring torque formed by the internal forces exerted by the part AC over the section C.

These forces arise as a result of elastic reaction against the extension of filaments over the neutral Surface and Compression below the neutral Surface. The magnitude of the restoring torque is YI/ρ. Where Y is the young's modulus of the material of the beam, I the geometrical moment of inertia of the section C about the neutral Surface and ρ the radius of curvature of the bent beam at C. At equilibrium,

$$W(l-x) = \frac{YI}{\rho}$$

$$\text{or} \quad \rho = \frac{YI}{W(l-x)} \qquad \text{——— (1)}$$

This expression shows that the radius of Curvature at a point of the beam varies inversely as $(l-x)$, the distance of the point from the loaded end. Thus in this case the beam is not bent in a circular arc.

Now, let y be the depression at the point C. Taking the end A as origin, let us draw x, y axes. Then (x,y) are the Co-ordinates of the point C, and the radius of Curvature at this point is given by,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\frac{d^2y}{dx^2}}$$

where $\frac{dy}{dx}$ is the slope of the tangent at a point (x,y). If the depression be within elastic limit, the slope will be Small. Therefore $(dy/dx)^2$ will be negligible compared to 1, and we Can put

$$\rho = \frac{1}{d^2y/dx^2}$$

Substituting for ρ in eqn ①

$$\frac{1}{\frac{d^2y}{dx^2}} = \frac{YI}{w(l-x)}$$

or, $$\frac{d^2y}{dx^2} = \frac{w}{YI}(l-x)$$

Integrating both sides

$$\frac{dy}{dx} = \frac{w}{YI}\int(l-x)\,dx$$

$$\frac{dy}{dx} = \frac{w}{YI}\left(lx - \frac{x^2}{2}\right) + A_1$$

where A_1 is constant of Integration. At the clamped end A of the beam the tangent is horizontal, i.e. at $x=0$, $\frac{dy}{dx}=0$. Substituting these values in the last expression $A_1=0$

$$\therefore \frac{dy}{dx} = \frac{w}{YI}\left(lx - \frac{x^2}{2}\right)$$

Integrating again.

$$y = \frac{w}{YI}\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) + A_2$$

Where A_2 is again a constant of integration. Again at $x=0$, we have $y=0$. so that $A_2=0$

$$\therefore y = \frac{w}{YI}\left(l\frac{x^2}{2} - \frac{x^3}{6}\right)$$

At B, we have $x=l$ and the depression y is a maximum. Let it be equal to δ. Then Substituting l and δ for x and y respectively in this above expression, we have

$$\delta = \frac{W}{YI}\left(\frac{l^3}{2} - \frac{l^3}{6}\right)$$

$$\delta = \frac{W}{YI}\left(\frac{3l^3 - l^3}{6}\right)$$

$$\delta = \frac{W}{YI} \times \frac{2l^3}{6}$$

$$\delta = \frac{Wl^3}{3YI} \quad \text{———} \quad (2)$$

If the beam is of rectangular cross-section, $I = \frac{bd^3}{12}$, where b is the breadth and d is the thikness of the beam. Therefore eqn (2)

$$\delta = \frac{Wl^3}{3Y\frac{bd^3}{\cancel{12}\,4}}$$

$$\delta = \frac{4Wl^3}{Ybd^3} \quad \text{———} \quad (3)$$

If the beam be of circular cross-section of radus r, then $I = \frac{\pi r^4}{4}$ then from (2)

$$\delta = \frac{Wl^3}{3Y\frac{\pi r^4}{4}}$$

$$\delta = \frac{4Wl^3}{3Y\pi r^4} \quad \text{———} \quad (4)$$

Equation (3) and (4) are the required expressions.